Calculus I

Problem Set II

April 22, 2025

Provide **handwritten** answers on a separate sheet of paper, and separate your challenge problems from the textbook problems. Typed answers will not be accepted. For full credit correct answers should be clear, legible, include explanations for your reasoning, and show all relevant work. You are allowed to make use of outside resources, including the internet, and friends, but you must cite your sources.

Textbook Problems:

Ch. 3: 116-125, 133-136,150-153, 159, 175-184, 209-213, 245-252, 257, 268-273, 296, 297,299 , 310-315, 320,325, 338-345

Challenge Problems

i) In this problem we explore the derivative of the common trigonometric functions.

a) Draw the graph of $\sin x$. By estimating the slope of the tangent line at each point, draw a graph of it's derivative. Do the same for $\cos x$. What do you notice?

b) Using the limits you calculated in the previous challenge homework, and the limit definition of the derivative to show that:

$$\frac{d}{dx}\sin x = \cos x \qquad \text{and} \qquad \frac{d}{dx}\cos x = -\sin x$$

c) Using the quotient rule find the derivatives of $\tan x$, $\cot x$, $\sec x$ and $\csc x$.

For *a*) just draw some graphs, and estimate the slope of the tangent lines. Everyone got points so long as they tried here. For *b*), recall from the previous challenge homework that:

$$\lim_{h\to 0}\frac{\sin h}{h} = 1 \qquad \text{and} \qquad \lim_{h\to 0}\frac{\cos h - 1}{h} = 0$$

We will also need the following results regarding angle sum identities:

$$\sin(x+h) = \sin x \cdot \cos h + \cos x \cdot \sin h$$
$$\cos(x+h) = \cos x \cdot \cos h - \sin h \sin x$$

The limit definition of the derivative tells us that:

$$\frac{d}{dx}\sin x = \lim_{h\to 0}\frac{\sin(x+h) - \sin x}{h}$$

Using the first identity we get:

$$\frac{d}{dx}\sin x = \lim_{h\to 0}\frac{\sin x\cdot\cos h + \cos x\cdot\sin h - \sin x}{h}$$

We regroup the terms to get:

$$\begin{aligned}\frac{d}{dx}\sin x &= \lim_{h\to 0}\frac{\sin x\cos h - \sin x + \cos x\sin h}{h}\\ &= \lim_{h\to 0}\left(\frac{\sin x(\cos h - 1)}{h} + \frac{\cos x\sin h}{h}\right)\end{aligned}$$

Now note that since:

$$\lim_{h\to 0}\frac{\sin x(\cos h-1)}{h} = \sin x\cdot\lim_{h\to 0}\frac{\cos h - 1}{h} = \sin x\cdot 0 = 0$$

and:

$$\lim_{h\to 0}\frac{\cos x\sin h}{h} = \cos x\cdot\lim_{h\to 0}\frac{\sin h}{h} = \cos x\cdot 1 = \cos x$$

we have that the addition rule for limits implies:

$$\begin{aligned}\frac{d}{dx}\sin x = \lim_{h\to 0}\frac{\sin x(\cos h-1)}{h} + \lim_{h\to 0}\frac{\cos x\sin h}{h}\\ =&0 + \cos x\\ =&\cos x\end{aligned}$$

as desired. For $\cos x$ we have that:

$$\frac{d}{dx}\cos x = \lim_{h\to 0}\frac{\cos(x+h) - \cos x}{h}$$

which by our second angle sum identity gives:

$$\frac{d}{dx}\cos x = \lim_{h\to 0}\frac{\cos x\cos h - \sin x\sin h - \cos x}{h}$$

Rearranging terms we obtain:

$$\begin{aligned}\frac{d}{dx}\cos x &= \lim_{h\to 0}\frac{\cos x\cos h - \cos x - \sin x\sin h}{h}\\ &= \lim_{h\to 0}\left(\frac{\cos x(\cos h - 1)}{h} - \frac{\sin h\sin x}{x}\right)\end{aligned}$$

Now since:

$$\lim_{h\to 0}\frac{\sin x\sin h}{h} = \sin x\cdot\lim_{h\to 0}\frac{\sin h}{h} = \sin x\cdot 1 = \sin x$$

and:

$$\lim_{h\to 0}\frac{\cos x(\cos h-1)}{h} = \cos x\cdot\lim_{h\to 0}\frac{\cos h - 1}{h} = \cos x\cdot 0 = 0$$

we have that by the limit rule for differences:

$$\begin{aligned}\frac{d}{dx}\cos x = \lim_{h\to 0}\left(\frac{\cos x(\cos h-1)}{h}\right) - \lim_{h\to 0}\left(\frac{\sin h\sin x}{x}\right)\\ =&0 - \sin x\\ =&-\sin x\end{aligned}$$

as desired.

For c) we need to apply the quotient rule. Recall that:

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g \cdot \frac{df}{dx} - f \cdot \frac{dg}{dx}}{g^2}$$

For $\tan x$ we have that:

$$\begin{aligned}\frac{d}{dx}\tan x =& \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)\\ =& \frac{\cos x \cdot \cos x - \sin x(-\sin x)}{\cos^2 x}\\ =& \frac{\cos^2 x + \sin^2 x}{\cos^2 x}\\ =& \frac{1}{\cos^2 x}\\ =& \sec^2 x\end{aligned}$$

For $\cot x$ we have that:

$$\begin{aligned}\frac{d}{dx}\cot x =& \frac{d}{dx}\left(\frac{\cos x}{\sin^2 x}\right)\\ =& \frac{\sin x \cdot (-\sin x) - \cos x \cos x}{\sin^2 x}\\ =& \frac{-\sin^2 x - \cos^2 x}{\sin^2 x}\\ =& \frac{-1}{\sin^2 x}\\ =& -\csc^2 x\end{aligned}$$

For $\csc x$:

$$\begin{aligned}\frac{d}{dx}\csc x =& \frac{d}{dx}\left(\frac{1}{\sin x}\right)\\ =& \frac{0 - \cos x}{\sin^2 x}\\ =& -\frac{\cos x}{\sin x} \cdot \frac{1}{\sin x}\\ =& -\cot x \csc x\end{aligned}$$

Finally, for $\sec x$:

$$\begin{aligned}\frac{d}{dx}\sec x =& \frac{d}{dx}\left(\frac{1}{\cos x}\right)\\ =& \frac{0 - (-\sin x)}{\cos^2 x}\\ =& \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x}\\ =& \tan x \sec x\end{aligned}$$

ii) In mathematics, sometimes we don't want to write out all the terms in a sum, so we employ something called sigma summation notation. This works as follows, say I have a function of integers, i.e. for each positive integer n, I have a new number $a(n)$. If I want to sum all of these numbers from 0 to some big N, we have to write:

$$a(0) + a(1) + a(2) + a(3) + \cdots + a(N-2) + a(N-1) + a(N)$$

Since mathematicians are lazy, we have decide that instead of writing this out every time, we should just write:

$$\sum_{n=0}^{n} a(n) = a(0) + a(1) + a(2) + a(3) + \cdots + a(N-2) + a(N-1) + a(N)$$

Concretely, if $a(n) = n$, and $N = 5$ then:

$$\sum_{n=0}^{5} n = 0 + 1 + 2 + 3 + 4 + 5 = 15$$

Now, we can also use this to write down polynomials. For example:

$$\sum_{n}^{N} a(n) \cdot x^n = a(0) + a(1)x + a(2)x^2 + a(3)x^3 + \cdots + a(N-2)x^{N-2} + a(N-1)x^{N-1} + a(N)$$

With $a(n) = n$ and $N = 5$ again we have that:

$$\sum_{n=0}^{5} nx^n = x + 2x^2 + 3x^3 + 4x^4 + 5x^5$$

Some functions can be represented as *infinite sums* of polynomials. Indeed, we have that:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{1.1}$$

by which we mean that if we plug a number, say 1 into e^x, then $e^1 = e$ is given by the infinite sum:

$$\sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{2} + \frac{1}{6} + \cdots$$

Now in practice we can't sum up infinitely many things, so what this actually means is that as we increase the number of terms in our sum, we get closer and closer to e.

a) By using Desmos to graph $\sum_{n=0}^{N}(x^n/n!)$ for larger and larger N, convince yourself that e^x is really given by (1.1). Do the same for:

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \qquad \text{and} \qquad \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

b) Using the power rule, and differentiate each infinite polynomial term by term to obtain the derivatives of e^x, $\sin x$ and $\cos x$.

c) Recall that i is an imaginary number satisfying $i^2 = -1$. Using the summation formulas for $\sin x$, $\cos x$, and e^x, show that:

$$e^{ix} = \cos x + i \sin x$$

Deduce that:

$$e^{i\pi} + 1 = 0$$

For a), there is no need to do anything, everyone got one point for saying they graphed these on Desmos.

For b), we have that:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \cdots$$

so we have that:

$$\begin{aligned}
\frac{d}{dx}e^x =& \frac{d}{dx}1 + \frac{d}{dx}x + \frac{d}{dx}\frac{x^2}{2} + \frac{d}{dx}\frac{x^3}{3!} + \frac{d}{dx}\frac{x^4}{4!} + \frac{d}{dx}\frac{x^5}{5!} + \frac{d}{dx}\frac{x^6}{6!} + \cdots \\
=& 0 + 1 + \frac{2x}{2} + \frac{3x^2}{3!} + \frac{4x^3}{4!} + \frac{5x^4}{5!} + \frac{6x^5}{6!} + \frac{7x^6}{6!} + \cdots \\
=& 0 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \cdots \\
=& e^x
\end{aligned}$$

as desired. For the trigonometric functions we have that:

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \cdots$$

while:

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \cdots$$

Taking a derivative of $\sin x$ we have that:

$$\begin{aligned}
\frac{d}{dx}\sin x =& \frac{d}{dx}x - \frac{d}{dx}\frac{x^3}{3!} + \frac{d}{dx}\frac{x^5}{5!} - \frac{d}{dx}\frac{x^7}{7!} + \frac{d}{dx}\frac{x^9}{9!} - \frac{d}{dx}\frac{x^{11}}{11!} + \cdots \\
=& 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \frac{7x^6}{7!} + \frac{9x^8}{9!} - \frac{11x^{10}}{11!} + \cdots \\
=& 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \cdots \\
=& \cos x
\end{aligned}$$

as desired. Taking a derivative of $\cos x$ we have that:

$$\begin{aligned}
\frac{d}{dx}\cos x & \frac{d}{dx}1 - \frac{d}{dx}\frac{x^2}{2} + \frac{d}{dx}\frac{x^4}{4!} - \frac{d}{dx}\frac{x^6}{6!} + \frac{d}{dx}\frac{x^8}{8!} - \frac{d}{dx}\frac{x^{10}}{10!} + \cdots \\
=& 0 - \frac{2x}{2} + \frac{4x^3}{4!} - \frac{6x^5}{6!} + \frac{8x^7}{8!} - \frac{10x^9}{10!} + \frac{12x^{11}}{12!} + \cdots \\
=& -x + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \frac{x^9}{9!} + \frac{x^{11}}{11!} + \cdots \\
=& -\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \cdots\right) \\
=& -\sin x
\end{aligned}$$

as desired. For c), we have to first show that $e^{ix} = \cos x + i \sin x$. Using the infinite sum definition of e^x, we know that:

$$e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = 1 + ix + \frac{(ix)^2}{2} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \cdots$$

We first note that $i^2 = -1$, so $i^3 = -i$, and $i^4 = 1$. It follows that:

$$e^{ix} = 1 + ix - \frac{x^2}{2} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \frac{x^6}{6!} + \cdots$$

We want to group together all the terms with i and all the terms without i, to get that:

$$\begin{aligned} e^{ix} &= \left(1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right) \\ &= \cos x + i \sin x \end{aligned}$$

as desired. Plugging in π for x we get:

$$e^{i\pi} = \cos\pi + i\sin\pi = -1 + i \cdot 0 = -1$$

hence we obtain Euler's identity:

$$e^{i\pi} + 1 = 0$$

often called one of the most beautiful equations in mathematics.

iii) In this exercise we confirm some statements made in lecture.

a) Using the chain rule, and product rule, derive the quotient rule by writing f/g as $f \cdot (g)^{-1}$.

b) Suppose that f is differentiable at a point a. Show that f is continuos at this point by showing $\lim_{x \to a} f(x) = f(a)$.

For *a*), we have that:

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{d}{dx}(f \cdot (g)^{-1})$$

The product rule tells us that:

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{df}{dx} \cdot \frac{1}{g} + f \cdot \frac{d(g)^{-1}}{dx}$$

The chain rule and the power rule tell us that:

$$\begin{aligned} \frac{d(g)^{-1}}{dx} &= \frac{d(g)^{-1}}{dg} \cdot \frac{dg}{dx} \\ &= \frac{-1}{g^2 \cdot} \frac{dg}{dx} \end{aligned}$$

hence:

$$\begin{aligned} \frac{d}{dx}\left(\frac{f}{g}\right) &= \frac{df}{dx} \cdot \frac{1}{g} + f \cdot \frac{-1}{g^2 \cdot} \frac{dg}{dx} \\ &= \frac{df}{dx} \cdot \frac{g}{g^2} - \frac{f}{g^2 \cdot} \frac{dg}{dx} \\ &= \frac{g \cdot \frac{df}{dx} - f \cdot \frac{dg}{dx}}{g^2} \end{aligned}$$

For b), we recall that if f is differentiable at a, we have that the following limit exists:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

If we set $h = x - a$, and look at the limit as x approaches a we obtain the same result written slightly differently:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

For all $x \neq a$ we have that:

$$f(x) - f(a) = \frac{f(x) - f(a)}{x - a} \cdot (x - a)$$

Since the limit as x approaches a of $x - a$ is equal to zero, we have that by the multiplication rule for limits:

$$\begin{aligned} \lim_{x \to a} f(x) - f(a) = &\lim_{x \to a} \frac{f(x) - f(a)}{x - a} \cdot \lim_{x \to a} (x - a) \\ =&f'(a) \cdot 0 \\ =&0 \end{aligned}$$

Since $f(a)$ is constant, we have that:

$$\lim_{x \to a} f(x) - f(a) = \left(\lim_{x \to a} f(x)\right) - f(a) = 0$$

hence:

$$\lim_{x \to a} f(x) = f(a)$$

so f is a continuous function.

iv) Find the the derivatives of the following functions:

$a)\ \log_a x \qquad b)\ \arcsin x$

$c)\ \arccos x \qquad d)\ \arctan x$

$e)\ \sinh x \qquad f)\ \cosh x$

Hint: $a)$ requires the chain rule, $b) - d)$ require inverse function theorem, and $e) - f)$ require only the derivative of e^x.

For $a)$ we have that

$$\frac{d}{dx} \ln x = \frac{1}{x}$$

Now what about $\log_a x$ for some real number $a > 0$? Well:

$$\log_a x = \frac{\ln x}{\ln a}$$

so taking a derivative gives us:

$$\frac{d}{dx}\log_a x = \frac{1}{x \cdot \ln a}$$

For b), if $-1 < x < 1$, we have that we can write x as $\sin\theta$ for some θ in the interval $(-\pi/2, \pi/2)$. It follows that:

$$\left(\frac{d}{dx}\arcsin\right)(\sin\theta) = \frac{1}{\cos(\theta)}$$

Now since $\sin\theta = x$, and:

$$\sin^2\theta + \cos^2\theta = 1 \tag{1.2}$$

we must have that:

$$\cos\theta = \sqrt{1 - \sin^2\theta} = \sqrt{1 - x^2}$$

hence:

$$\frac{d}{dx}\arcsin = \frac{1}{\sqrt{1 - x^2}}$$

For c) if $-1 < a < 1$ we have that we can write x as $\cos\theta$ for some θ in the interval $(-\pi, \pi)$. The inverse function theorem gives us:

$$\left(\frac{d}{dx}\arccos\right)(\cos\theta) = \frac{1}{-\sin\theta}$$

Now what is $-\sin\theta$? By the same argument as in the previous example, we have that:

$$\sin^2\theta + \cos^2\theta = 1$$

hence:

$$\sin\theta = \sqrt{1 - \cos^2\theta} = \sqrt{1 - x^2}$$

it follows that:

$$-\sin\theta = -\sqrt{1 - x^2}$$

Therefore:

$$\frac{d}{dx}\arccos x = \frac{-1}{\sqrt{1 - x^2}}$$

For d), we have that for any x we can write it as $\tan\theta$ for some θ in the interval $(-\pi/2, \pi/2)$. We can thus write:

$$\left(\frac{d}{dx}\arctan\right)(\tan\theta) = \frac{1}{\sec^2\theta}$$

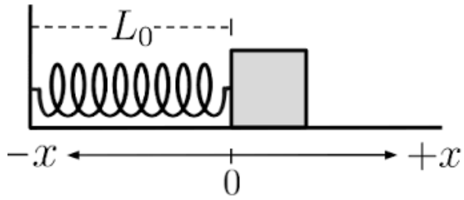
Now, dividing equation $\sin^2\theta + \cos^2\theta = 1$ by $\cos^2\theta$ gives us:

$$\tan^2\theta + 1 = \sec^2\theta$$

hence we have that $\sec^2\theta = 1 + x^2$ as $x = \tan\theta$. It follows that:

$$\frac{d}{dx}\arctan x = \frac{1}{1 + x^2}$$

v) In this problem we consider a block of mass m attached to a spring which is then attached to a wall. When the spring is not compressed at all, it has a length of L_0, in particular we have the following picture:



If we stretch the string a distance of A in the positive x direction, and then let it go, results from physics tell us that the boxes distance from the $x = 0$ position is given as a function of time by:

$$x(t) = A \cdot \cos\left(t \cdot \sqrt{\frac{k}{m}}\right)$$

Here k is the spring constant which measures how hard it is to pull or push the string.

a) Find the velocity function $v(t)$.

b) Given any velocity function, $v(t)$, argue that the derivative of $v(t)$ should be an acceleration function, telling us the instantaneous acceleration at an time. Find $a(t)$ in this situation.

c) Newton's laws state that the force function is given by $F(t) = m \cdot a(t)$. Hooke's law states that the force function for a spring is given by $F(t) = -k \cdot x(t)$. Verify that $m \cdot a(t) = -k \cdot x(t)$,[1] i.e. that that the distance function of the box satisfies Hooke's Law.

For *a*), to find the velocity function $v(t)$ we need to take a derivative of $x(t)$ with respect to t. We have that:

$$v(t) = \frac{dx}{dt} = \frac{d}{dt}\left[A \cdot \cos\left(t \cdot \sqrt{\frac{k}{m}}\right)\right]$$

We can pull the A because it a constant, hence we need only use the chain rule

$$\begin{aligned}\frac{d}{dt}\left[\cos\left(t \cdot \sqrt{\frac{k}{m}}\right)\right] &= \frac{d\cos\left(t\sqrt{k/m}\right)}{d(t\sqrt{k/m})} \cdot \frac{d(t\sqrt{k/m})}{dt} \\ &= -\sin\left(t\sqrt{\frac{k}{m}}\right) \cdot \sqrt{\frac{k}{m}} \\ &= -\sqrt{\frac{k}{m}}\sin\left(t \cdot \sqrt{\frac{k}{m}}\right)\end{aligned}$$

It follows that:

$$v(t) = -A \cdot \sqrt{\frac{k}{m}}\sin\left(t \cdot \sqrt{\frac{k}{m}}\right)$$

For *b*), the derivative of a velocity function will tell us how the velocity is changing at any point with respect to time. In particular, dv/dt has units of meters per second per second, i.e. m/s^2.

[1]To do this just find $m \cdot a(t)$, and $-k \cdot x(t)$ and see the are exactly the same.

This implies that dv/dt is our acceleration function, because acceleration tells us how velocity is changing at any, just as velocity tells us how the position is changing at any point. We now take the derivative of $v(t)$ via the same method as before:

$$\begin{aligned}
a(t) =& \frac{dv}{dt} \\
=& -A \cdot \sqrt{\frac{k}{m}} \cdot \frac{d}{dt}\left[\sin\left(t \cdot \sqrt{\frac{k}{m}}\right)\right] \\
=& -A \cdot \sqrt{\frac{k}{m}} \cdot \frac{d\sin\left(t\sqrt{k/m}\right)}{d(t\sqrt{k/m})} \cdot \frac{d(t\sqrt{k/m})}{dt} \\
=& -A \cdot \sqrt{\frac{k}{m}} \cdot \cos\left(t \cdot \sqrt{\frac{k}{m}}\right) \cdot \sqrt{\frac{k}{m}} \\
=& -A \cdot \frac{k}{m} \cdot \cos\left(t \cdot \sqrt{\frac{k}{m}}\right)
\end{aligned}$$

For c), we see that:

$$\begin{aligned}
F(t) =& m \cdot a(t) \\
=& -m \cdot A \cdot \frac{k}{m} \cdot \cos\left(t \cdot \sqrt{\frac{k}{m}}\right) \\
=& -k \cdot A \cdot \cos\left(t \cdot \sqrt{\frac{k}{m}}\right) \\
=& -k \cdot x(t)
\end{aligned}$$

hence Hooke's law holds in this situation, as expected.