# Calculus I

## Challenge Homework Set IV

May 8, 2025

Provide **handwritten** answers on a separate sheet of paper. Typed answers will not be accepted. For full credit correct answers should be clear, legible, include explanations for your reasoning, and show all relevant work. You are allowed to make use of outside resources, including the internet, and friends, but you must cite your sources. **Textbook Problems**:

Ch 4: 1-4, 12-20, 32-38

i) A conical tank (vertex down) has height $H$ and base radius $R$. Water leaks out at constant rate

$$\frac{dV_{\text{leak}}}{dt} = k$$

while at the same time water is pumped in at a rate of:

$$\frac{dV_{\text{in}}}{dt} = \alpha h^2$$

where $\alpha > 0$ is a constant, and $h$ is the height of the water in the tank.

*a*) Express the volume the water of the tank as a function of $h$. **Hint:** use similar triangles.

*b*) Find an expression for $dh/dt$. **Hint:** what should $dV/dt$ be in terms of $dV_{\text{in}}/dt$ and $dV_{\text{leak}}/dt$?

*c*) Suppose that $k = 2\,\text{m}^3\,/\,\text{min}$, $\alpha = 3\,\text{m}\,/\,\text{min}$, $H = 12\,\text{m}$, and $R = 6\,\text{m}$. If at $t = 3$ minutes we have $h = 6\,\text{m}$, evaluate and interpret $dh/dt$.

For *a*), we have that volume of water in the cone is $V = \frac{1}{3}\pi r^2 h$. Since $r$ and $h$ and $R$ and $H$ make a right triangle with the same angles, we have that:

$$\frac{r}{h} = \frac{R}{H} \Rightarrow r = \frac{Rh}{H}$$

It follows that:

$$V(h) = \frac{\pi R^2 h^3}{3H^2} \tag{1.1}$$

For *b*) we have that $dV/dt$ is equal to the rate at which water is pumped in minus the rate at which water leaks. It follows that:

$$\frac{dV}{dt} = \alpha h^2 - k$$

Taking a derivative of (1.1) we have that:

$$\frac{dV}{dt} = \frac{\pi R^2 h^2}{H^2}\frac{dh}{dt}$$

Hence we have that:

$$\frac{dh}{dt} = \frac{(\alpha h^2 - k)H^2}{\pi R^2 h^2}$$

For $c$) we plug in all our values to find that:

$$\frac{dh}{dt} = \frac{(3 \cdot 6^2 - 2) \cdot 12^2}{\pi \cdot 6^2 \cdot 6^2} = \frac{106}{9\pi} \approx 3.75\,\mathrm{m}\,/\,\mathrm{min}$$

which means that $t = 3$ min the height of the cone is changing at a rate of 3.75 meters per minute.

ii) A hemispherical bowl of radius $R$ sits open-side up. Sand of density $\rho$ is poured so that at time $t$ the sand occupies a spherical cap of height $h(t)$. The mass of sand in the dome increases at constant rate

$$\frac{dM}{dt} = k$$

Note that the volume the spherical cap is given by:

$$V(h) = \frac{\pi h^2}{3}(3R - h) \tag{1.2}$$

$a$) Find $M(h)$ in terms of $h$, $R$, and $\rho$.

$b$) Find an expression for $dh/dt$ in terms of $k$, $h$, $R$ and $\rho$.

$c$) If $R = 2\,\mathrm{m}$, $\rho = 1000\,\mathrm{kg}\,/\,\mathrm{m}^3$, $k = 6\,\mathrm{kg}\,/\,\mathrm{min}$, find $dh/dt$ when $h = 1$.

For $a$), we have that $M$ is density times volume hence:

$$M(h) = \frac{\rho\pi h^2}{3}(3R - h) = \frac{\rho\pi}{3}(3Rh^2 - h^3)$$

For $b$) we take a time derivative to obtain:

$$\begin{aligned}\frac{dM}{dt} &= \frac{dM}{dh} \cdot \frac{dh}{dt} \\ &= \frac{\rho\pi}{3}(6Rh - 3h^2) \cdot \frac{dh}{dt} \\ &= \rho\pi(2Rh - h^2)\frac{dh}{dt}\end{aligned}$$

Since $dM/dt = k$ we also have that:

$$k = \frac{\rho\pi}{3}(6Rh - 3h^2) \cdot \frac{dh}{dt}$$

solving fro $dh/dt$:

$$\frac{dh}{dt} = \frac{k}{\rho\pi(2Rh - h^2)}$$

plugging every in we obtain that when $h = 1$:

$$\frac{dh}{dt} = \frac{6}{1000\pi(4-1)} = \frac{6}{1000\pi \cdot 3} = \frac{1}{500\pi}\,\mathrm{m}\,/\,\mathrm{min}$$

which is approximately .0006 meters per minute.

iii) A spherical weather balloon contains $n$ moles of gas; the ideal gas law states that:

$$PV = nRT$$

where $P$ is pressure (measured in pascals, Pa), $V$ is the volume of the gas, $T$ is the temperature of the gas (measured in Kelvin, K), and $R$ is the Boltzman constant.

*a*) What are the units of $R$?

*b*) As the weather balloon rises, the temperature changes, pressure, and volume (assume the ballon can stretch so that it's radius can change) of the gas change. If $r$ is the radius of the ballon as a function of time, find an expression relating $dr/dt$, $dP/dt$ and $dT/dt$.

*c*) Suppose that after rising for 45 minutes, we have that $T = 300\,\mathrm{K}$, $P = 1 \times 10^5\,\mathrm{Pa}$, $r = 1/2\,\mathrm{m}$, $dr/dt = 1/10\,\mathrm{m}\,/\,\mathrm{min}$, and $dP/dt = -1/10\,\mathrm{Pa}\,/\,\mathrm{min}$. Assuming that $R = 1$, and $n = 1\,\mathrm{mol}$ (this is nonphysical but makes the calculations easier), find and interpret $dT/dt$ at 45 minutes.

The units of $R$ are given by $\mathrm{Pa} \cdot \mathrm{m}^3\,/(\mathrm{mol} \cdot \mathrm{K})$. In words this pascals meters cubed per mole kelvin.

We take a time derivative of $PV = nRT$ to get:

$$\frac{dP}{dt}V + P\frac{dV}{dt} = nR \cdot \frac{dT}{dt}$$

Now $V = 4/3\pi r^3$, hence $dV/dt = 4\pi r^2 \frac{dr}{dt}$ giving us:

$$\frac{dP}{dt} \cdot \frac{4\pi r^3}{3} + 4P\pi r^2 \frac{dr}{dt} = nR\frac{dT}{dt}$$

We plug in every number to get that:

$$\frac{dT}{dt} = -\frac{4\pi}{3 \cdot 10 \cdot 8} + \frac{4 \cdot 100000 \cdot \pi}{4 \cdot 10} = 3.14 \times 10^4\,\mathrm{K}\,/\,\mathrm{min}$$