Calculus I

Challenge Homework Set I

May 16, 2025

Provide **handwritten** answers on a separate sheet of paper. Typed answers will not be accepted. For full credit correct answers should be clear, legible, include explanations for your reasoning, and show all relevant work. You are allowed to make use of outside resources, including the internet, and friends, but you must cite your sources. **Textbook Problems**:

Ch 4: 113-128, 316, 322, 338-340

i) Find the critical points of the following functions, and evaluate whether the critical points are maxima or minima.

   a) $\cos(2x)$

   b) $\sin(5x)$

   c) $\sin(|x|)$

   d) $|\cos x|$

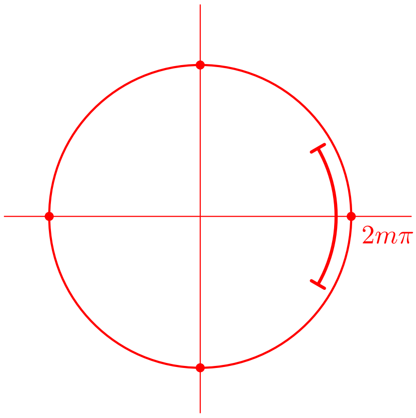
   For *a*), we have that:

$$\frac{d\cos(2x)}{x} = -2\sin(2x)$$

   This is zero precisely when $2x = n\pi$ where n is some integer. In other words, our critical points are of the form $n\pi/2$. If n is even, we have that $n = 2m$ for some other integer m; in particular, we have that $x = m\pi$. We choose a value of x slightly greater than $m\pi$, say $m\pi + .1$ then we get:

$$-2\sin(2m\pi + .2)$$

   which is negative as $\sin(2m\pi + .2)$ is positive sin $2m\pi + .2$ lies in the following area on the unit circle:

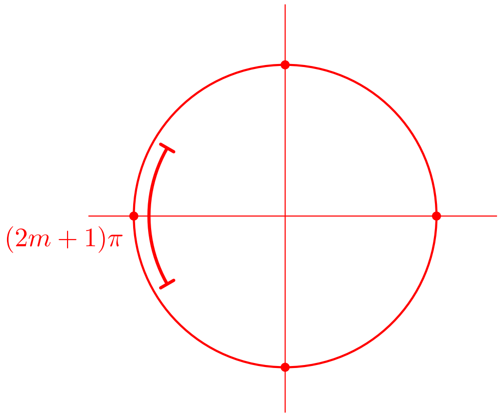


Similarly, if we chose a value of x slightly less than $m\pi$, say $m\pi - .1$ wend up with $-\sin(2m\pi - .2)$. Since this angle ends up below the x-axis we have that $-\sin(2m\pi - .2)$ is positive, and so $n\pi/2$ is a local maximum when n is even.

If n is odd, we have that $n = 2m + 1$, hence $x = (2m + 1)\pi/2 = m\pi + \pi/2$. If we chose a value of x which is slightly larger than $m\pi + \pi/2$, say $m\pi + \pi/2 + .1$, then we get:

$$-2\sin(2m\pi + \pi + .2)$$

We see that $2m\pi + \pi + .2$ lies in the following area on the unit circle:



In particular, $2m\pi + \pi + .2$ gives us an angle below the x axis, and so $-2\sin(2m\pi + \pi + .2)$ is positive. Similarly, we have that $-2\sin(2m\pi + \pi - .2)$ is negative, hence when n is odd we have that $n\pi/2$ is a local minimum. We thus conclude that the minimums occur when n is odd, and the maximums occur when n is even.

For b), we have that:

$$\frac{d\sin(5x)}{dx} = 5\cos(5x)$$

This is zero when $\cos(5x) = 0$, which implies that $5x = n\pi + \pi/2$ for every integer n. Our critical points are then of the form:

$$x = \frac{n\pi}{5} + \frac{\pi}{10}$$

If n is even, then we have that $n = 2m$ for some integer m, hence:

$$x = \frac{2m\pi}{5} + \frac{\pi}{10}$$

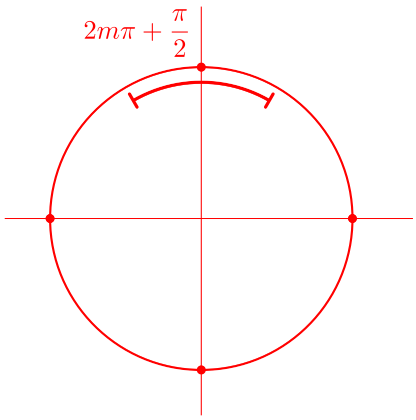
If we choose a number slightly larger than the above, say:

$$a = \frac{2m\pi}{5} + \frac{\pi}{10} + .1$$

then:

$$5a = 2m\pi + \frac{\pi}{2} + .5$$

which lies in the following area of the unit circle:



It follows that:

$$\cos\left(2m\pi + \frac{\pi}{2} + .5\right) < 0$$

as $2m\pi + \frac{\pi}{2} + .5$ lies to the left of the y-axis. Similarly, if we choose a number slightly less than x say:

$$b = \frac{2m\pi}{5} + \frac{\pi}{10} - .1$$

then same argument demonstrates that $\cos(5b) > 0$ hence $n\pi/5 + \pi/10$ is a local maximum when n is even.

Suppose that n is odd then:

$$x = \frac{(2m+1)\pi}{5} + \frac{\pi}{10}$$

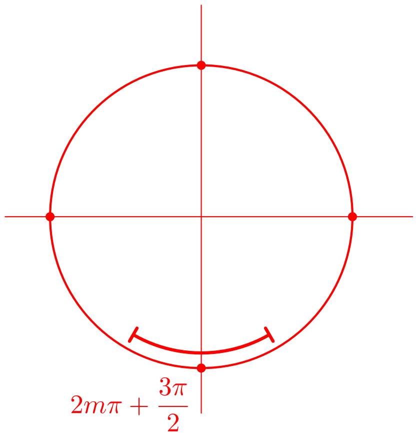
If we choose a number greater than x say:

$$a = \frac{(2m+1)\pi}{5} + \frac{\pi}{10} + .1$$

then:

$$5a = (2m+1)\pi + \frac{\pi}{2} + .5$$

lays in the following area of the unit circle:



It follows that $\cos(5a) > 0$, and if we set:

$$b = \frac{(2m+1)\pi}{5} + \frac{\pi}{10} - .1$$

then:

$$5b = (2m+1)\pi + \frac{\pi}{2} - .5$$

so $\cos(5b) < 0$. It follows that if n is odd then $n\pi/5 + \pi/10$ is a local minimum.

For c), we have that since $\sin x$ is odd:

$$\sin|x| = \begin{cases} -\sin x & \text{for } x < 0 \\ \sin x & \text{for } x \geq 0 \end{cases}$$

Taking a derivative we have that:

$$\frac{d\sin|x|}{dx} = \begin{cases} -\cos x & \text{for } x < 0 \\ \cos x & \text{for } x > 0 \end{cases}$$

Note that since $-\cos(0) = -1 \neq 1 = \cos(0)$ we have that 0 is a critical point as the derivative can't exist there. Moreover, we know that 0 is a local minimum as at 0, we have that $d\sin|x|/dx$ goes from negative to positive.

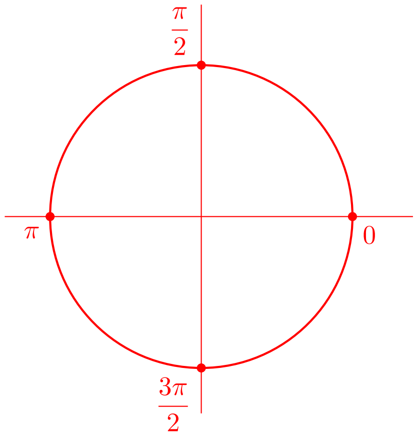
For $x > 0$, we have that critical points are $x = n\pi + \pi/2$ where m is a positive integer. By Example 6.10, we know that here the critical points are local maximums when n is even, and local minimums when n is odd.

For $x < 0$, we have that critical points are also of the form $n\pi + \pi/2$ where this time m is a negative integer. Here we deduce that the critical points are the opposite as to the case of Example 6.10 as when x is less than zero, we are finding the local minima and maxima of $-\cos x$. In other words, since we are multiplying by negative, local minima become local maxima and vice versa. It follows that for $x < 0$ the critical points are local maxima when n is odd, and local minima when n is even.

For d), we have that:

$$|\cos x| = \begin{cases} -\cos x & \text{for } \cos x < 0 \\ \cos x & \text{for } \cos x > 0 \end{cases}$$

But when is $\cos x$ less than zero, and when is it greater than zero? Well examining the unit circle, we see that:



so we have that $\cos x > 0$ when $2n\pi - \pi/2 \leq x \leq 2n\pi + \pi/2$, and $\cos x < 0$ when $2n\pi + \pi/2 \leq x \leq 2n\pi - \pi/2$. It follows that:

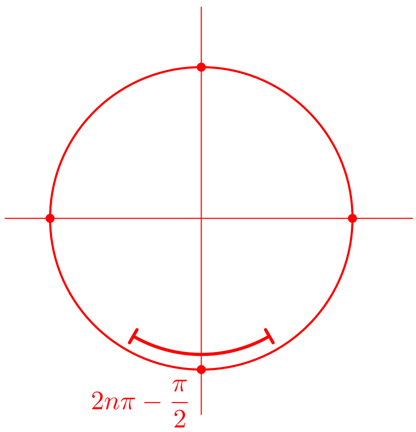
$$|\cos x| = \begin{cases} \cos x & \text{for } 2n\pi - \pi/2 \leq x \leq 2n\pi + \pi/2 \\ -\cos x & \text{for } 2n\pi + \pi/2 \leq x \leq 2n\pi - \pi/2 \end{cases}$$

Taking a derivative, we obtain that:

$$\frac{d|\cos x|}{dx} = \begin{cases} -\sin x & \text{for } 2n\pi - \pi/2 < x < 2n\pi + \pi/2 \\ \sin x & \text{for } 2n\pi + \pi/2 < x < (2n+1)\pi - \pi/2 \end{cases}$$

Note that for all n $2n\pi - \pi/2$ and $2n\pi + \pi/2$, $\sin x$ and $-\sin x$ differ by a minus sign so the derivative at $2n\pi - \pi/2$ and $2n\pi + \pi/2$ does not exist for all integers n. It follows that for all n the points $2n\pi - \pi/2$ and $2n\pi + \pi/2$ are critical points. Our other critical points occur when $\sin x = 0$ or $-\sin x = 0$, which are when $x = m\pi$ for an integer m.

For critical points of the form $2n\pi - \pi/2$, we are on the in the following area of the unit circle:



hence for $a < 2n\pi - \pi/2$, we have that $\sin(a)$ is negative, and when $a > 2n\pi - \pi/2$ we have that $-\sin(a)$ is positive so $2n\pi - \pi/2$ is a local minimum. Similarly for $2n\pi + \pi/2$ we are at the top of the unit circle, so for $a < 2n\pi + \pi/2$ we have that $-\sin(a)$ is negative, and for $a > 2n\pi + \pi/2$ $\sin(a)$ is positive, implying that $2n\pi + \pi/2$ is also a local minimum.

When m is odd we have that $x = (2l+1)\pi$ which lies in the interval $[2l\pi + \pi/2, (2l+1)\pi - \pi/2]$. In this interval we are looking at the extrema $-\cos x$, and since m is odd we have that this is a local maximum by the argument in part c). When m is even, we have that $x = 2l\pi$, which lies in the interval $[2l\pi - \pi/2, 2l\pi + \pi/2]$ which means we are looking at the local extrema of $\cos x$. By Example 6.10 in the notes, we have that here x is a local maximum as well.

It follows that local maximums occurs when $x = n\pi$ for every integer n, and local minimums occur when $x = 2n\pi \pm \pi/2$ for every integer n.

ii) In this problem we consider optimizing the volume or surface area of certain shapes. **Hint:** Draw pictures!

a) Find the largest volume of a cylinder that fits into a cone of radius r and height h.

b) Find the dimensions of a cylinder with volume $16\pi\,\mathrm{m}^2$ that has the least surface area.

c) Find the dimensions of a right cone with surface area $4\pi\,\mathrm{m}^2$ that has the largest volume.

d) Suppose that total surface area of a cube and and sphere is $1\,\mathrm{m}^3$. Find the dimensions of the cube and sphere such that the total volume is maximized.

For a), we let the radius of the cylinder be R and its height H. We have that a cylinder sitting inside the cone cuts out a cone of height $h - H$ and radius equal to R. By exploiting similar triangle we obtain the following relation:

$$\frac{R}{h-H} = \frac{r}{h}$$

Solving for H we obtain that:

$$H = \frac{(r-R)h}{r}$$

It follows that the volume of our cylinder is:

$$V = \frac{\pi h}{r}R^2 \cdot (r - R) = \frac{\pi h}{r}(rR^2 - R^3)$$

Taking a derivative:

$$\frac{dV}{dr} = \frac{\pi h}{r}(2rR - 3R^2)$$

Setting this equal to zero gives the following equation:

$$2rR - 3R^2 = 0$$

which implies that either $R = 0$ or $R = 2r/3$. Clearly $R = 0$ can't be a maximum, and if $R < 2r/3$ then $V(R) > 0$, while if $R > 2r/3$ we have that $V(R) < 0$, hence $2r/3$ is a local maximum. This is the only critical point which is not an endpoint on the interval $[0, r]$, hence $2r/3$ is a global maximum. Plugging $2r/3$ into the volume function we get that the maximum volume is:

$$V = \frac{4\pi r^2 h}{27}$$

For b) we have that if a cylinder has a volume of 16π, then:

$$\pi r^2 h = 16\pi \Rightarrow h = \frac{16}{r^2}$$

The surface area is given by:

$$S = 2\pi rh + 2\pi r^2 = \frac{32\pi}{r} + 2\pi r^2$$

Taking a derivative we get that:

$$\frac{dS}{dr} = -\frac{32\pi}{r^2} + 4\pi r$$

Setting this equal to zero is the same as solving:

$$\frac{32\pi}{r^2} = 4\pi r \Rightarrow 4r^3 = 32 \Rightarrow r^3 = 8 \Rightarrow r = 2\,\mathrm{m}$$

This is a local minimum as if r is less than 2 then dS/dr is less than zero, and if r is greater than 2 then dS/d is greater than zero. Since $r = 2$ is the only critical it follows that $r = 2$ must be a global minimum. Since $r = 2$ implies $h = 4$, we have that these are dimensions which minimize the volume when surface area is 16π.

For c), if $l = \sqrt{r^2 + h^2}$, then the surface area of a cone is given by:

$$\pi r\sqrt{r^2 + h^2} + \pi r^2 = 4\pi$$

implying that:

$$r\sqrt{r^2 + h^2} + r^2 = 4 \Rightarrow r\sqrt{r^2 + h^2} = 4 - r^2$$

Squaring both sides gives:

$$r^2(r^2 + h^2) = (4 - r^2)^2$$

hence:

$$r^2 + h^2 = \frac{(4 - r^2)^2}{r^2} \Rightarrow h = \sqrt{\frac{(4 - r^2)^2}{r^2} - r^2}$$

Note that:

$$\frac{(4 - r^2)^2}{r^2} - r^2 = \frac{(4 - r^2)^2 - r^4}{r^2} = \frac{16 - 8r^2}{r^2}$$

hence:

$$h = \sqrt{\frac{16}{r^2} - 8}$$

The volume function is given by:

$$V = \pi r^2\sqrt{\frac{16}{r^2} - 8} = 2\pi\sqrt{4r^2 - 2r^4}$$

This is maximized when $4r^2 - 2r^4$ is maximized, and this had derivative:

$$8r - 8r^3$$

hence $r = 0, 1$ are critical points. When r is less than 1 but greater than zero, we have that $8r - r^3$ is positive, and for r greater than 1 we have that $8r - r^3$ is negative. It follows that $r = 1$ is a local maximum, and is in fact a global maximum on the interval $[0, 1]$, hence the cone has the highest volume when $r = 1$, which also forces $h = 2\sqrt{2}$ as well.

For d), we have that the surface area of a cube of side length s is $6s^2$, while the surface area of the sphere is $4\pi r^2$. The volumes are s^3, and $4/3\pi r^3$ respectively. If:

$$4\pi r^2 + 6s^2 = 1$$

we have that:

$$r^2 = \frac{1-6s^2}{4\pi}$$

hence:

$$r^3 = \frac{(1-6s^2)^{3/2}}{8\pi\sqrt{\pi}}$$

so our volume can be written as a function of s:

$$V = s^3 + \frac{(1-6s^2)^{3/2}}{6\sqrt{\pi}}$$

Taking a derivative, we get:

$$\begin{aligned}\frac{dV}{ds} =& 3s^2 + \frac{1}{6\sqrt{\pi}} \cdot \left(\frac{3}{2}(1-6s^2)^{1/2} \cdot (-12s)\right) \\ =& 3s^2 - \frac{3s\sqrt{1-6s^2}}{\sqrt{\pi}}\end{aligned}$$

Finding critical points is then the same as solving:

$$3s^2 = \frac{3s\sqrt{1-6s^2}}{\sqrt{\pi}}$$

Clearly $s = 0$ is a critical point, but what are the others? Assuming $s \neq 0$, we can divide by s and find that:

$$s = \frac{\sqrt{1-6s^2}}{\sqrt{\pi}}$$

squaring both sides yields:

$$\pi s^2 = 1 - 6s^2 \Rightarrow (\pi + 6)s^2 = 1 \Rightarrow s = \frac{1}{\sqrt{\pi + 6}}$$

Our interval is $[0, 1/\sqrt{6}]$ as those are the allowed values of s. By comparing values, (either using a calculator or otherwise), one easily sees that $s = 0$ is our maximum value. When $s = 0$, the radius $1/\sqrt{4\pi}$, and these dimensions maximize volume.

iii) For this problem recall that (x_1, y_1) and (x_2, y_2) are two points in the plane, then the distance between them is given by:

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Using this, answer the following questions:

- Where is the line $y = 5 - 2x$ closest to the origin?

- Where is the parabola $y = x^2$ closest to the point $(2, 0)$?

- Where is the cubic $y = x^3$ closest to the point $(2, 2)$?

Note that in all of these it suffices to maximize d^2 instead of x.

For a), we have that:

$$d^2 = (x - 0)^2 + (y - 0)^2 = x^2 + (5 - 2x)^2$$

Taking a derivative we get:

$$\frac{dd^2}{dx} = 2x - 2 \cdot 2(5 - 2x) = 2x - 20 + 8x = 10x - 20$$

The critical points of d^2 in this case occurs when $x = 2$. We see that this clearly a local minimum, and it is a global minimum because $x = 2$ is the only critical point. By plugging 2 back into our equation for y, we find that the point $(2, 1)$ is the point on the line closest to the origin.

For b), we have that:

$$d^2 = (x - 2)^2 + (y - 0)^2 = (x - 2)^2 + x^4$$

Taking a derivative we get that:

$$\frac{dd^2}{dx} = 2(x - 2) + 4x^3 = 4x^3 + 2x - 2$$

This part can only be done with calculator, so my apologies for this problem. In particular, $x \approx .589$ is a critical point of d^2, and this then ends up being global minimum for d^2.

For c), we have that:

$$d^2 = (x - 2)^2 + (y - 2)^2 = (x - 2)^2 + (x^3 - 2)^2$$

Taking a derivative, we get that:

$$\frac{dd^2}{dx} = 2(x - 2) + 6x^2(x^3 - 2) = 6x^5 - 12x^2 + 2x - 2$$

This one can also only be done with a calculator, I am again sorry, this is actually a problem from the textbook, and I thought it would work out nicer. In the problem $x \approx 1.248$ is a critical point, and it also ends up being a global minimum.

iv) An object with mass m is dragged along a horizontal plane by a force acting along a rope attached to the object. If the rope makes an angle θ with a plane, then the magnitude of the force is:

$$F = \frac{\mu m g}{\mu \sin\theta + \cos\theta}$$

where g is the acceleration due to gravity, and μ is a dimensionless constant called the coefficient of friction. For what value of θ is F minimized?

We need to take a derivative of F with respect to θ. Note that physically, θ is constrained to the interval $[0, \pi/2]$. We take a derivative of F with respect to θ:

$$\frac{dF}{d\theta} = \mu m g \cdot \frac{\sin\theta - \mu\cos\theta}{(\mu\sin\theta + \cos\theta)^2}$$

Note that in the interval $[0, \pi/2]$, the denominator is never zero as both $\sin\theta$ and $\cos\theta$ are positive. It follows that critical points occur when:

$$\sin\theta - \mu\cos\theta = 0 \Rightarrow \tan\theta = \mu$$

It follows that $\tan^{-1}(\mu)$ is our critical point, and since $\tan\theta$ is an increasing function on $[0, \pi/2]$ we know that for $\theta < \tan^{-1}(\mu)$ we must have that $\tan(\theta) < \mu$ hence and for $\tan^{-1}(\mu) < \theta$ we have that $\mu < \tan\theta$, hence $dF/d\theta$ goes from negative to positive at $\tan^{-1}(\mu)$ and so $\tan^{-1}(\mu)$ is a global minimum on $[0, \pi/2]$.