Midterm II Study Guide

MA 251

Problems are roughly ordered from simpler to harder within each section — if you're tight on time, start from the top and work your way down. The exam won't include all of these, but anything on here is fair game. In particular, the exam will have more 'easy' questions than I have included here, so if you can do all of these consistently you will be more than prepared.

Related Rates

Functions with Multiple Dependent Variables

Make sure you're comfortable with the following ideas:

- The basic derivatives, and the differentiation rules. You won't be explicitly tested on these as you were in midterm one, but you will have to take derivatives.
- Using the chain rule, and product rule for multiple functions depending on a single variable.
- Given an equation relating multiple dependent variables you should be comfortable finding an equation which relates the rates of change/derivatives of the dependent variables.

Example Problems

(1) Let x and y depend on t, and satisfy:

$$x^2 + y^2 = 4$$

Find an equation relating dx/dt and dy/dt.

(2) Let w, y, and z depend on x, and satisfy:

$$wyz = \sin(z)$$

Find an equation relating dw/dx, dy/dx, and dz/dx.

(3) Let x, y, and z depend on w and satisfy:

$$\arctan(x^2y^2z^2) = e$$

Find an equation relating dx/dw, dy/dw, and dz/dw.

(4) Suppose that x and y depend on t, and satisfy:

$$tx^2 + \sin(ty) = t^2$$

Find an equation relating dx/dt and dy/dt.

(5) If $x^2 + y^2 + z^2 = 9$, $dx/dt = 5$, and $dy/dt = 4$, find dz/dt when $(x, y, z) = (2, 2, 1)$.

(6) Suppose that x and y depend on t, and that when $x = 2$, and $y = 1$ we have that $dx/dt = -1$. If:

$$e^{xy} = e^2$$

find dy/dt.

(7) If:

$$\sin(x)y = \cos(w)z$$

$dx/dt = 1$, $dy/dt = -2$, and $dw/dt = 1$, find dz/dt when $(x, y, w, z) = (-\pi/4, -1, \pi/4, 1)$.

(8) If:

$$(x^2 + 4)\sin y = e^w z^2.$$

and

$$\frac{dx}{dt} = 2, \qquad \frac{dy}{dt} = -3, \qquad \frac{dw}{dt} = \frac{1}{2},$$

find $\frac{dz}{dt}$ at the instant

$$(x, y, w, z) = \left(1,\ \tfrac{\pi}{2},\ 0,\ \sqrt{5}\right).$$

(9) If:

$$e^{x\,y}\cos w = z\big(1 + \ln z\big).$$

and

$$\frac{dx}{dt} = -1, \qquad \frac{dy}{dt} = 3, \qquad \frac{dw}{dt} = 0.5,$$

find $\frac{dz}{dt}$ when

$$(x, y, w, z) = \big(0,\ 2,\ 0,\ 1\big).$$

(10) If:

$$e^{x\,y} + \ln z = \arctan w.$$

and:

$$\frac{dx}{dt} = 2, \qquad \frac{dy}{dt} = -1, \qquad \frac{dw}{dt} = 3,$$

find $\frac{dz}{dt}$ at the point

$$(x, y, w, z) = \left(0,\ 2,\ \sqrt{3},\ e^{\pi/3-1}\right).$$

Modeling Problems

Make sure you can do the following:

- Read word problems carefully and extract the information needed to solve them.
- Draw pictures which represent the problem (you won't be explicitly tested on this, but it will be helpful!)
- Physically interpret the derivative as a rate of change, depending on the units of the problem.
- Set up equations which mathematically represent the problem.
- Use calculus, and algebra to solve for the unknown rate of change.

Example Problems

(1) What is the formula for the area of a triangle, the area of a square, the area of a circle, the volume of a rectangular prism, the volume of cylinder, the volume of a cone, the volume of a sphere, the distance between points (x_1, y_1) and (x_2, y_2).

(2) Each side of a square is increasing at a rate of 6 cm/s. At what rate is the *area* of the square increasing when the area of the square is 16 cm^2?

(3) The length of a rectangle is increasing at a rate of 8 cm/s and its width is increasing at a rate of 3 cm/s. When the length is 20 cm and the width is 10 cm, how fast is the area of the rectangle increasing?

(4) The length of a rectangle is increasing at a rate of 8 cm/s and its width is increasing at a rate of 3 cm/s. When the length is 20 cm and the width is 10 cm, how fast is the area of the rectangle increasing?

(5) At noon, ship A is 100 km *west* of ship B. Ship A sails due south at 35 km/h while ship B sails due north at 25 km/h. How fast is the distance between the two ships changing at 4:00 PM?

(6) An object moves along the curve

$$y = 2\sin\left(\frac{\pi x}{2}\right).$$

As the particle passes through the point $\left(\frac{1}{3}, 1\right)$, its x-coordinate increases at a rate of $\sqrt{10}$ cm/s. How fast is the distance from the particle to the point $(0, 0)$ changing at that instant?

(7) Water leaks out of an inverted right circular conical tank at 10,000 cm^3/min while water is simultaneously pumped *into* the tank at a constant rate. The tank is 6 m tall and 4 m across at the top. If the water level is rising at 20 cm/min when the depth of the water is 2 m, find the rate (in cm^3/min) at which water is being pumped into the tank.

(8) A weather balloon is released from level ground 500m away from a tracking station. The balloon rises vertically at a constant speed of 3m/s. Let θ be the angle of elevation as measured from the tracking station, find $d\theta/dt$ when the boon is 200m off the ground.

(9) If the minute hand of a clock has length r (in centimeters), find the rate at which it sweeps out area as a function of r.

(10) The minute hand on a watch is 8 mm long and the hour hand is 4 mm long. How fast is the distance between the tips of the hands changing at one o'clock?

Optimization and the Second Derivative

Minima and Maxima

Make sure you can do the following:

- Define critical points, local minimums/maximums, and global minimums/maximums.
- Find the critical points of function graphically.
- Find the critical points of a function mathematically.
- Determine on what intervals a function is increasing, or decreasing, and use this to identify critical points as local minima and maxima.
- Find the global minima and maxima of a function on a closed interval.

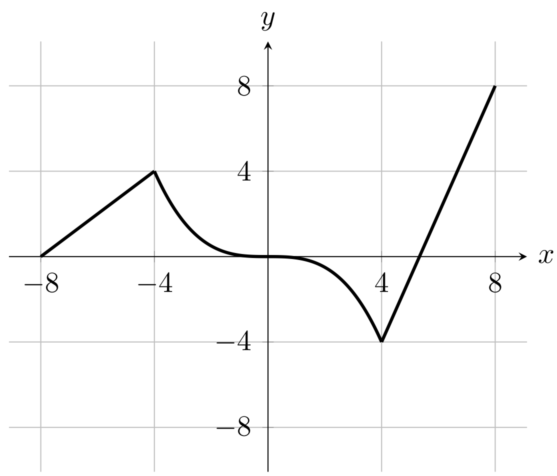
Example Problems

(1) What is a critical point?

(2) When is a critical point a local minimum? A local maximum?

(3) What is a global minimum/maximum? What is the difference between a local minimum/maximum and a global minimum/maximum?

(4) Consider the following graph of $f(x)$ defined on the interval $[-8, 8]$:



(a) What are the critical points of f?

(b) On what intervals is f increasing? How about decreasing?

(c) Which of these critical points are local minimums? Which are local maximums?

(d) Which points are global minimums? Which points are global maximums?

(5) Draw a function defined on $[-10, 10]$ which has critical points at -4, 1, and 7. One of the critical points should be a local minimum, one should be a local maximum, and one should be neither.

(6) If a function as two critical points, is it possible for both of them to be maximums? Why or why not?

(7) What are the critical points of $\sin x$? Which of these critical points are local minimums, and which are local maximums?

(8) What are the critical points of $\cos x$? Which of these critical points are local minimums, and which are local maximums?

(9) What are the critical points of $-|x| + 3$? Which are local minimums, and which are local maximums?

(10) Find the local minima and maxima of the following functions:

(a) $x^2 - 3x + 2$

(b) $x^3 - 6x^2$

(c) $x^3 + 6x^2 + x + 12$

(d) $\frac{x^4}{2} + 3x^3 + x^2 - 7$

(11) Find the global minima and maxima of the following functions:

(a) $12 + 4x - x^2$ on the interval $[0, 5]$.

(b) $2x^3 - 3x^2 - 12x + 1$ on the interval $[-2, 3]$.

(c) $(x^2 - 4)^3$ on the interval $[-2, 3]$.

(d) $x + \frac{1}{x}$ on the interval $[1/5, 4]$.

(e) $\frac{\sqrt{x}}{1+x^2}$ on the interval $[0, 2]$

(f) $2\cos x + \sin 2x$ on the interval $[0, \pi/2]$.

(g) $x + \cos(x/2)$ on the interval $[\pi/4, 7\pi/4]$

(h) $\ln(x^2 + x + 1)$ on the interval $[-1, 1]$.

(i) $x - 2\arctan x$ on the interval $[0, 4]$.

The Second Derivative

Make sure you can do the following:

- Physically, and mathematically interpret the second derivative.
- Determine on what intervals a function is concave up or concave down.
- Find the tangent parabola to a function.
- Use the second derivative to identify local minima and maxima.

Example Problems

(1) What does it mean for a function to be concave up, or concave down at a point? How does this concept relate to the second derivative?

(2) What is an inflection point of a function?

(3) Refer to graph of $f(x)$ in problem (**4**) on the previous section. On what intervals is f concave up? On what intervals is f concave down? Are any points in the interval $[-8, 8]$ inflection points?

(4) Draw a function which is concave down for $x < 0$ and concave for $x > 0$. Does this function have an inflection point?

(5) if $r(t)$ describes the distance of a rocket ship from the earth in kilometers as a function of seconds, what does the second derivative describe? In particular what are the units on the second derivative?

(6) Find the second derivative of the following functions:

(a) $\sin x$

(b) $\sec x$

(c) $e^{\cos x}$

(d) $\arctan x$

(e) $\cos x^2$

(f) $\cos(x)$.

(7) Find the tangent parabola of the following functions at the prescribed points:

(a) $\cos x$ at $x = \pi/4$

(b) $\sec x$ at $x = \pi/4$

(c) $e^{\cos x}$ at $x = \pi/2$

(d) $\arctan x$ at $x = 1$

(e) e^{x^2} at $x = 0$

(f) $\cos^2(e^x)$ at $x = \ln(\pi/4)$

(8) What is the second derivative test? When does it fail to work? Why does it fail in this situation?

(9) Redo problems **(10)** and **(11)** but use the second derivative test.

Optimization Problems

Make sure you can do the following:

- Read word problems carefully and extract the information needed to solve them.
- Draw pictures which represent the problem (you won't be explicitly tested on this, but it will be helpful!)
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- Set up equations which mathematically represent the problem.
- Use calculus, and algebra to solve the problem.

Example Problems

(1) Find two numbers whose difference is 100 and whose product is a minimum.

(2) Find two positive numbers whose product is 100 and whose sum is a minimum.

(3) The sum of two positive numbers is 16. What is the smallest possible value of the sum of their squares?

(4) What is the maximum vertical distance between the line

$$y = x + 2$$

and the parabola

$$y = x^2$$

for $-1 \leq x \leq 2$?

(5) What is the minimum vertical distance between the parabolas

$$y = x^2 + 1,$$
$$y = x - x^2$$

(6) Find the dimensions of a rectangle with perimeter 100 m whose area is as large as possible.

(7) A farmer wants to fence in a rectangular plot of land adjacent to the north wall of his barn. No fencing is needed along the barn, and the fencing along the west side of the plot is shared with a neighbor who will split the cost of that portion of the fence. If the fencing costs \$20 per linear foot to install and the farmer is not willing to spend more than \$5000, find the dimensions for the plot that would enclose the most area.

(8) Find the largest volume of a cylinder that fits into a cone of radius r and height h.

(9) Find the dimensions of a cylinder with volume $16\pi\,\mathrm{m}^2$ that has the largest surface area.

(10) Find the dimensions of a right cone with surface area $4\pi\,\mathrm{m}^2$ that has the largest volume.

(11) Suppose that total surface area of a cube and and sphere is $1\,\mathrm{m}^3$. Find the dimensions of the cube and sphere such that the total volume is maximized.

(12) Find the point on the line $y = 2x13$ that is closest to the origin.

(13) If the two equal sides of an isosceles triangle have length a, find the length of the third side that maximizes the area of the triangle.

(14) A piece of wire 10 m long is cut into two pieces. One piece is bent into a square and the other is bent into an equilateral triangle. How should the wire be cut so that the total area enclosed is a maximum? A minimum?

(15) A right circular cylinder is inscribed in a sphere of radius r. Find the largest possible surface area of such a cylinder