Midterm I Study Guide

MA 251

Problems are roughly ordered from simpler to harder within each section — if you're tight on time, start from the top and work your way down. The exam won't include all of these, but anything on here is fair game. In particular, the exam will have more 'easy' questions than I have included here, so if you can do all of these consistently you will be more than prepared.

Limits

Long Term Limits

Make sure you're familiar with the following ideas:

- Interpreting $\lim\limits_{x\to\pm\infty} f(x)$ as the long term behavior of a function.
- Using big number logic to analyze limits of $x \to \pm\infty$ of elementary functions.
- Understanding what it means for $\lim\limits_{x\to\pm\infty} f(x)$ to be equal to L, $\pm\infty$, or not exist.

Example Problems

(1) Evaluate the limit:

$$\lim_{x\to-\infty} \frac{x^5+7x^2+3}{x^4-2x^3+x}$$

(2) Evaluate the limit:

$$\lim_{x\to\infty} \frac{17x^2+2x-1}{x^5-x^2+3x}$$

(3) Evaluate the limit:

$$\lim_{x\to-\infty} \frac{17x^5+2x-1}{x^5-x^2+3x}$$

(4) Evaluate the limits:

$$\lim_{x\to\infty} 2^{-x} \qquad \text{and} \qquad \lim_{x\to-\infty} 2^{-x}$$

(5) Evaluate the limits:

$$\lim_{x\to\infty} \cos xe^{x} \qquad \text{and} \qquad \lim_{x\to-\infty} \sin xe^{x}$$

(6) Evaluate the limit:

$$\lim_{x \to \infty} \frac{e^x}{\ln x}$$

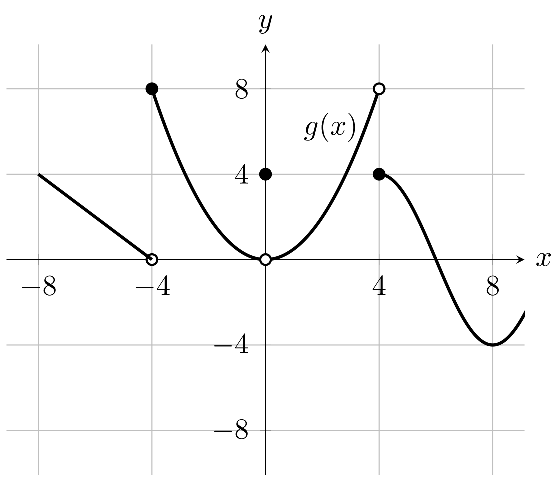
Limits at a Point and Continuity

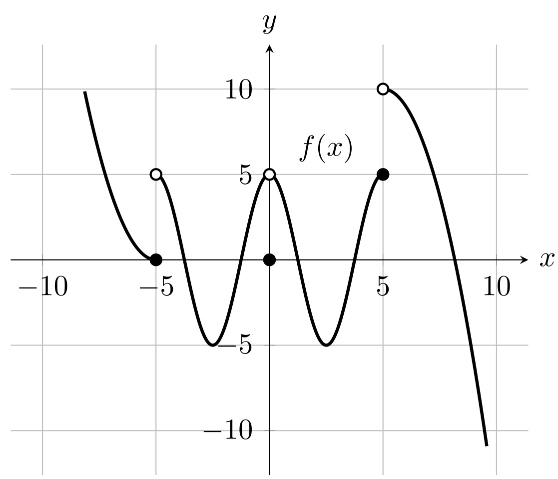
Make sure you can do the following:

- Interpret $\lim_{x \to a} f(x)$ as the behavior of f around $x = a$; i.e. what is $f(x)$ approaching as x approaches a?
- Explain left and right handed limits.
- Be able to argue whether $\lim_{x \to a} f(x)$ is a real number L, $\pm\infty$, or does not exist, by analyzing the left and right handed limits.
- Evaluate basic limits using algebraic techniques, and the limit laws.
- Analyze limits by looking at the given graph of a function.
- Explain what it means for a function to be continuous in terms of limits.
- Be able to recognize different types of discontinuities, i.e. removable, infinite, and jump.

Example Problems

For the following questions please make use of the following graphs:





(1) Find the following limits:

$a)\ \lim_{x\to 0} f(x)$	$b)\ \lim_{x\to 0} g(x)$	$c)\ \lim_{x\to 0} (f\cdot g)(x)$
$d)\ \lim_{x\to -4^-} g(x)$	$e)\ \lim_{x\to -4^+} g(x)$	$f)\ \lim_{x\to -4} g(x)$
$g)\ \lim_{x\to 4^-} g(x)$	$h)\ \lim_{x\to 4^+} g(x)$	$i)\ \lim_{x\to 4} g(x)$
$j)\ \lim_{x\to -4^-} f(x)$	$k)\ \lim_{x\to -4^+} f(x)$	$l)\ \lim_{x\to -4} f(x)$
$m)\ \lim_{x\to 4^-} f(x)$	$n)\ \lim_{x\to 4^+} f(x)$	$o)\ \lim_{x\to 4} f(x)$

(2) Is f continuous at 0? Is g? What about $(f\cdot g)$? If any are not continuous, describe their discontinuity types.

(3) Is f continuous at -4? Is g? If any are not continuous, describe their discontinuity types.

(4) Is f continuous at 4? Is g? If any are not continuous, describe their discontinuity types.

(5) Find the limit of:

$$\lim_{x\to 2} \frac{2^x}{\sin(\pi x/4)}$$

Is this function continuous at 2? If not what is it's discontinuity type?

(6) Find the following limits:

$$\lim_{x\to x} \frac{1}{x} \qquad \text{and} \qquad \lim_{x\to 0} \frac{1}{x^2}$$

Are the above functions continuous at zero? If not, what are their discontinuity types?

(7) Find the following limit:

$$\lim_{x\to 2} \frac{x^2-4}{\sqrt{12-x^3}-2}$$

(8) Find the following limit:

$$\lim_{x\to -1} \frac{x+1}{\sqrt{2-x^2}-1}$$

(9) Find a value for k so that the following piecewise function is continuous at $x=0$:

$$f(x) = \begin{cases} e^x & \text{for } x \le 0 \\ \sin(kx)\text{for}x \ge 0 \end{cases}$$

(10) Find a value for k so that the following function continuous at $x = 1$:

$$f(x) = \begin{cases} \ln x & \text{for} x \leq 1 \\ \cos(kx) & \text{for} x \geq 1 \end{cases}$$

Derivatives

Definitions and Rules

Make sure you can do the following:

- Interpret the derivative physically as the instantaneous rate of change at a point, i.e. given a position function interpret the derivative as velocity, given a velocity function interpret the derivative as acceleration, given a population over time function interpret the derivative of as population growth rate.
- Interpret the derivative mathematically as the slope of the tangent line at a point.
- Given the graph of a function, determine where a function is differentiable.
- State the limit definition of the derivatives.
- Given certain limits, and trigonometric identities, or infinite sums, find the derivatives of the functions e^x, $\sin x$, and $\cos x$.
- State the derivatives of all the elementary functions.
- State the product rule, the quotient rule, the chain rule, and the inverse function theorem.

Example Problems

(1) State the limit definition of the derivative.

(2) Fill in the following table:

Function $f(x)$	**Derivative** $f'(x)$
c	
x^n	
e^x	

a^x	
$\ln x$	
$\log_a x$	
$\sin x$	
$\cos x$	
$\tan x$	
$\arcsin x$	
$\arccos x$	
$\arctan x$	
$\sinh x$	
$\cosh x$	

(3) Suppose that $T(r)$ is a function describing the temperature in fahrenheit of a room a distance r away from a fireplace. What does the derivative of T describe, i.e. $T(r)$ has units temperature, r has units meters, what are the units of $T'(r)$? If $T(2) = 80$, how do we physically interpret this? If $T'(2) = 5$, how do we physically interpret this?

(4) A pandemic is sweeping across the globe. Suppose that $I(t)$ is a function describing how many sick people there are t days after the start of the pandemic. What does $I'(t)$ describe? Interpret $I(300) = 3,000,000$, and $I'(30) = -200,000$ physically.

(5) Using the limit definition of the derivative, or an infinite sum and the power rule, determine the derivatives of the following functions:

$$a)\ e^x \qquad b)\ \sin x \qquad c)\ \cos x$$

You may use the following facts:

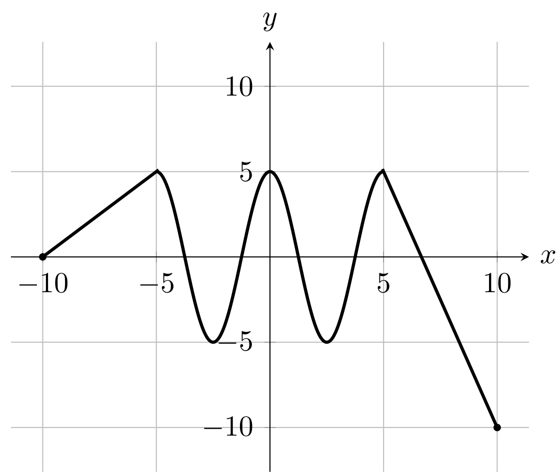
$$e^x = \sum_{n=0\infty} \frac{x^n}{n!} \qquad \sin x = \sum_{n=0} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \qquad \cos x = \sum_{n=0} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1 \qquad \lim_{h\to 0} \frac{\sin h}{h} = 1 \qquad \lim_{h\to 0} \frac{\cos x - 1}{x} = 0$$

as well as:

$$\sin(\theta+\phi) = \sin\theta\cos\phi + \sin\phi\cos\theta$$
$$\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$

(6) Examine the following graph of the function $f(x)$, defined on the interval $[-10, 10]$:



Where is $f(x)$ differentiable? Your answer should be in interval notation. For what values of x is is the derivative of f equal to zero? What is the derivative of f at $x = -8$? Sketch a graph of the derivative of $f(x)$.

(7) State the following rules: product rule, quotient rule, chain rule, and the inverse function theorem.

Calculations and Problem Solving

Make sure you can do the following:

- Recognize a function h as a sum, product, quotient, or composition of two functions f and g.
- Apply the power rule, chain rule, product rule, quotient rule, and sum rule as necessary to obtain the derivative of a function.
- Find the tangent line to a function at a point.
- Use the derivative to solve basic word problems.

(1) Take the derivative of the following functions:

$a)$ $\sin^2 x$	$b)$ $x^2 e^{\cos x}$	$c)$ $\arccos(\cos x)$
$d)$ $\ln(\sin x \cos x)$	$e)$ $\dfrac{e^x}{\sin x}$	$f)$ $\dfrac{e^{\tan x}}{\ln x}$
$g)$ $e^{\cos(\ln x)}$	$h)$ $\cos(e^{\arcsin x})$	$i)$ $\sin\left(\dfrac{x^2+4}{x^2-4}\right)$
$j)$ $\ln(\arctan x)$	$k)$ $\ln(\cosh x)$	$l)$ $e^{\cos x \cdot \sin^2 x + x^2}$
$m)$ $e^{e^{e^x}}$	$n)$ $e^{e^{e^{x}}}$	$o)$ $e^{e^{e^{e^x}}}$

(2) Let $f(x) = 3x^3 + x^2 - 1$. Find the line tangent to the graph f at $x = 1$.

(3) Let $f(x) = \sin x \cos x \tan x$. Find the line tangent to the graph at $x = \pi$.

(4) Let $f(x) = e^{\sin^2 x}$. Find the line tangent to the graph at $x = \pi/2$.

(5) Let $f(x) = \ln(\arctan x)$. Find the line tangent to the graph at $x = \sqrt{2}/2$.

(6) Let $f(x) = \sin(e^{x^2})$. Find the line tangent to the graph at $x = \sqrt{\ln(\pi/2)}$.

(7) Let $f(x) = \cos(\ln(x^3))$. Find the line tangent to the graph at $x = e^{\pi/3}$.

(8) A particle moves along a straight line, and its position at time t seconds is given by

$$s(t) = 3t^2 - 2t + 5 \quad \text{(in meters)}.$$

Find the velocity of the particle at $t = 4$. What are the units of your answer?

(9) The temperature (in °C) of a cooling object after t minutes is given by

$$T(t) = 100e^{-0.1t} + 20.$$

How fast is the temperature changing at $t = 5$? Is the object heating up or cooling down at that moment?

(10) The height (in meters) of a tree after t years is modeled by

$$h(t) = 4\ln(t^2 + 1).$$

How fast is the tree growing at $t = 3$? What are the units of your answer?

(11) A population of bacteria grows according to the function

$$P(t) = 1000e^{t/100}, \quad \text{where } t \text{ is in hours.}$$

How fast is the population growing at $t = 10$? What are the units of your answer?

(12) The side length (in meters) of a square at time t seconds is given by

$$s(t) = 2 + \sin(\pi t^2).$$

How fast is the area of the square changing at time $t = 1$? Include units and interpret your answer.

(13) The force (in newtons) required to move an object is a function of its distance from a magnetic source, given by

$$F(r) = \frac{5}{r^2 + 1},$$

where r is measured in meters. The object is being moved along a path so that its position at time r seconds is given by

$$r(t) = \sqrt{t^2 + 1}.$$

How fast is the force changing at time $t = 2$?

(14) The brightness (measured in Lumens) B of a star observed through a telescope is a function of its distance r from Earth, given by

$$B(r) = \frac{1}{r^2 + 1}.$$

Suppose the distance (in light-years) is increasing with time according to

$$r(t) = e^{t/100}, \quad \text{where } t \text{ is in years.}$$

How fast is the observed brightness changing at $t = 0$?

Implicit Differentiation and Logarithmic Differentiation

Make sure you can do the following:

- Differentiate both sides of an equation containing both y and x.
- Use the chain rule for y by recognizing y as a function of x.
- Solve for $\frac{dy}{dx}$ to obtain a derivative in terms of y and x.
- Find the tangent line to a point lying on the solution set to an equation containing both y and x.
- Recognize when you can use the natural log to take more complicated derivatives.

Example Problems

(1) Using implicit differentiation compute dy/dx for the following equations:

$a)\ x^2 + y^2 = 1$	$b)\ xy + \sin(y) = 1$	$c)\ x^2y^3 = \cos(x+y)$
$d)\ \ln(xy) + x^2 = y^2$	$e)\ e^{xy} = x + y$	$f)\ \tan(xy) = x^2 + y^2$
$g)\ \sin\left(e^{x+y^2}\right) = x^2 + y$	$h)\ \arcsin(xy) + x = y$	$i)\ \ln\left(\cos(xy)\right) = x + y$

(2) Find the line tangent to the graph of $x^2 + y^2 = 1$ at $(\sqrt{3}/2, 1/2)$.

(3) Find the line tangent to the graph of $xy + y^2 = 3$ at $(1, 1)$.

(4) Find the line tangent to the graph of $\sin(xy) = x + y$ at $(\pi/2, 0)$.

(5) Find the line tangent to the graph of $e^{xy} + x^2 = y$ at $(0, 1)$.

(6) Use logarithmic differentiation to find the following derivatives:

$$a)\ f(x) = x^x \qquad b)\ y = \frac{(x^2+1)^5 \cdot \sqrt{\sin x}}{(x^3-4)^2} \qquad c)\ f(x) = \left(\frac{\ln x}{x}\right)^x$$